

# ‘Strange’ occurrences in *SuperEnalotto*

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## Abstract

In this paper a way is suggested for calculating the probability of *consecutive numbers* strings within a sequence of  $n$  numbers randomly drawn (without replacement) among the set of the first  $N$  consecutive numbers, with  $N \gg n$ .

An explicit derivation is carried out for the special case of *SuperEnalotto*, nowadays the most famous lottery in Italy, with  $N = 90$  and  $n = 6$ . It turns out that, on average, one every three drawings presents one or more *consecutive numbers* strings inside.

## 1 Introduction

Among lotteries and gambling games endorsed by the Italian government, *SuperEnalotto* is surely the most popular one, even in the neighbor states like France, Austria and Switzerland. It has been introduced for the first time on December 3, 1997 and since then it has provided some of the biggest payoffs of all lotteries in the world [1].

The game is currently issued three times a week. The player gains the jackpot (which increases at each issue, depending on the total number of players each time and on the number of the previous not winning issues) if she matches the 6 numbers drawn out of 90 (from 1 to 90), regardless of the order in which the numbers are drawn.

Concerning the winning odds, *SuperEnalotto* is considered to be one of the most difficult gambling game among the existing ones: winning the jackpot means to match the drawn un-ordered collection of six numbers among all

the possible 6-combinations of 90 numbers. It is easy to see that there are  $\binom{90}{6}$  of these combinations, so the winning odds at every issue of the game are 1 in 622,614,630.

To give an idea of the tininess of the figures at play, the odds **you** have to gain the jackpot in the next issue of the game are comparable [2] with the odds **you** have to be hit, here on the Earth, by a kilometre-sized asteroid during the very same day you play the game! Obviously, this does not mean that no one will never gain the jackpot: many people play the game, thus the expectation value of winnings is not tiny at all.

Although the author is not a greedy player of *SuperEnalotto*<sup>1</sup>, his attention was drawn by an apparently counter-intuitive high frequency with which some peculiar number patterns appear within the six drawn numbers. In particular, he noted an unexpected high occurrence<sup>2</sup> of *consecutive numbers*, namely like those in the last drawings of year 2009 as reported in Table 1.

Among conspiracy-inclined people, and there is plenty of them between greedy players, this feature is read as the unequivocal sign that drawings are intentionally biased by the lottery operator.

On the contrary, the first thought of the author was to understand if the laws of probability are able to explain such apparently bizarre behavior. Obviously, they do and in the following Section it is shown how.

## 2 Consecutive numbers probability

First of all, let us define the *consecutive numbers* probability  $\mathcal{C}(n, N)$ : this is the probability to have one or more strings of consecutive numbers (of all lengths, starting from strings of 2 consecutive numbers up to strings of  $n$  consecutive numbers) within a sequence of  $n$  numbers randomly drawn from the set  $\{1, 2, 3, 4, \dots, N\}$  of the first  $N$  consecutive numbers. For the sake of simplicity, the *consecutive numbers* probability is explicitly derived here only for the special case with  $N = 90$ , and  $n = 6$ , namely the case of *SuperEnalotto*. Generalizations of such result should not pose particular difficulties.

In what follows every drawn sequence is thought as written with numbers in increasing order (as in Table 1) and in the following format:

$$s_1 a_1 s_2 a_2 s_3 a_3 s_4 a_4 s_5 a_5 s_6 a_6 s_7, \quad (1)$$

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<sup>1</sup>Probably because he is pretty confident that he will never be hit by a kilometre-sized asteroid in his whole life.

<sup>2</sup>Indeed, this is a personal impression, not an objective fact.

Date	Drawn numbers					
31/12/2009	<b>40</b>	<b>41</b>	45	<b>51</b>	<b>52</b>	79
29/12/2009	11	16	21	62	70	84
28/12/2009	34	36	<b>61</b>	<b>62</b>	78	88
24/12/2009	5	25	37	52	62	79
22/12/2009	6	13	17	52	64	82
19/12/2009	9	15	17	34	73	83
17/12/2009	1	23	32	53	66	83
15/12/2009	1	4	48	53	61	85
12/12/2009	10	16	40	55	66	77
10/12/2009	33	36	39	50	65	70
09/12/2009	9	30	50	57	62	70
05/12/2009	20	63	66	<b>74</b>	<b>75</b>	77
03/12/2009	19	30	34	49	54	74
01/12/2009	21	25	36	51	82	90
28/11/2009	3	42	52	67	78	86
26/11/2009	26	<b>31</b>	<b>32</b>	50	61	77
24/11/2009	19	47	69	71	<b>86</b>	<b>87</b>
21/11/2009	<b>36</b>	<b>37</b>	50	57	73	83
19/11/2009	1	35	42	55	57	62
17/11/2009	16	25	28	30	39	51
14/11/2009	15	27	31	40	63	82
12/11/2009	37	43	49	66	73	88
10/11/2009	24	32	56	59	76	81
07/11/2009	9	11	70	<b>74</b>	<b>75</b>	84
05/11/2009	12	20	28	57	74	85
03/11/2009	22	35	44	66	71	88
31/10/2009	13	16	41	49	58	76
29/10/2009	29	37	42	52	54	59
27/10/2009	38	<b>43</b>	<b>44</b>	<b>45</b>	63	77
24/10/2009	6	26	<b>37</b>	<b>38</b>	55	73
22/10/2009	29	<b>33</b>	<b>34</b>	<b>64</b>	<b>65</b>	87
20/10/2009	41	47	49	55	77	90
17/10/2009	8	24	45	47	72	84
15/10/2009	3	29	51	55	73	81
13/10/2009	5	11	40	43	56	82
10/10/2009	9	<b>33</b>	<b>34</b>	52	<b>82</b>	<b>83</b>
08/10/2009	1	12	17	65	70	90
06/10/2009	<b>23</b>	<b>24</b>	49	<b>69</b>	<b>70</b>	72
03/10/2009	28	38	<b>59</b>	<b>60</b>	71	83
01/10/2009	4	19	34	52	68	77

Table 1: *SupeEnalotto* drawings in the last three months of year 2009. In bold: *consecutive numbers* strings.

where the  $a_i$  are the drawn numbers (with  $a_i < a_j$ , if  $i < j$ ), while the  $s_k$  are numbers which express the distance between two contiguous  $a_i$ . Namely,  $s_2 = (a_2 - a_1) - 1$ ,  $s_3 = (a_3 - a_2) - 1$ , and so on, while  $s_1 = a_1 - 1$  and  $s_7 = (90 - a_6)$ . As a particular case  $s_1 = 0$ , if  $a_1 = 1$ , and  $s_7 = 0$ , if  $a_6 = 90$ .

An important arithmetical relation then holds:

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 = 90 - 6 = 84, \quad (2)$$

with  $0 \leq s_k \leq 84$ .

Equation (2) allows to cope with consecutive numbers within the drawn sequence: as a matter of fact, one has consecutive numbers strings when at least one of the value  $\{s_2, s_3, s_4, s_5, s_6\}$  is equal to zero. The values of  $s_1$  and  $s_7$  are not important for the consecutive numbers issue.

Thus, in order to calculate the numerical value of  $\mathcal{C}(6, 90)$ , one has to count all the ways of writing the integer 84 as a sum of the numbers  $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ , with at least one of the value  $\{s_2, s_3, s_4, s_5, s_6\}$  equal to zero: in fact,  $\mathcal{C}(6, 90)$  is equal to the ratio between this number of ways (let

call it  $N_c$ ) and the ways of writing 84 as the sum of the numbers  $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ , without any proviso on their values, namely, with  $0 \leq s_k \leq 84$ . Let call this last number  $N_t$ .

This is a partition problem and like most of partition problems it can be solved thanks to the technique of *generating functions*.

Consider first the calculation of the denominator of the above probability fraction, the number  $N_t$ .

Every  $s_k$  spans from 0 to 84; thus, let us expand the following polynomial power:

$$(1 + x + x^2 + x^3 + \dots + x^{82} + x^{83} + x^{84})^7 = \left( \frac{x^{85} - 1}{x - 1} \right)^7. \quad (3)$$

After the expansion, the multiplicative coefficient of the term  $x^{84}$  provides exactly the number  $N_t$ : such coefficient sums up the number of way in which the term  $x^{84}$  can be obtained as a product of the seven terms  $x^l$  (one for each  $s_k$ ), and thus, it sums up also the ways in which the exponent of  $x^{84}$  can be obtained as a sum of the seven exponents  $l$ , with  $0 \leq l \leq 84$ .

Applying multiple derivative on eq. (3) and making a suitable normalization (to simplify the numerical coefficient equal to  $84!$  which originates from the exponent of  $x^{84}$  after multiple derivative), a relatively simple algorithm for calculating the coefficient of  $x^{84}$  can be obtained as follows:

$$N_t = \frac{1}{84!} \cdot \frac{d^{84}}{dx^{84}} \left[ \left( \frac{x^{85} - 1}{x - 1} \right)^7 \right] \Big|_{x=0} = 622,614,630. \quad (4)$$

The numerical result in eq. (4) has been obtained in less than a minute with free math software **Sage** [3], running on a not so brand-new laptop. The reader must have recognized such figure. It is the total number of 6-combinations of the set of 90 numbers, namely  $\binom{90}{6}$ , as it should be.

The next step is to calculate the number  $N_c$ . This task is a bit more involved, although it does not require anything new with respect to the derivation of the number  $N_t$ . According to the format of the drawn sequence introduced in eq. (1), one has *consecutive numbers* when at least one of the five numbers  $\{s_2, s_3, s_4, s_5, s_6\}$  is equal to zero.

Let us start with the simplest case when only one  $s_k$  among  $\{s_2, s_3, s_4, s_5, s_6\}$  is equal to zero while the other ones are different from zero. This case corresponds to the situation in which there is only a single pair of *consecutive numbers* within the drawn sequence.

First of all, there are  $\binom{5}{1} = 5$  cases in which one such  $s_k$  may be equal to zero, since there are exactly five numbers  $s_k$ . Moreover, for each one of such

cases one must count the ways in which the relation (2) is fulfilled with one  $s_k$  equal to zero and the other ones assuming values  $1 \leq s_k \leq 84$ .

The arguments introduced above suggest that the this number of ways is equal to the coefficient of the term  $x^{84}$  in the expansion of the following polynomial powers product:

$$(1+x+x^2+\dots+x^{83}+x^{84})^2(x+x^2+\dots+x^{83}+x^{84})^4 = \left(\frac{x^{85}-1}{x-1}\right)^2 \left(\frac{x^{85}-1}{x-1}-1\right)^4, \quad (5)$$

where the factor to the left on both sides (that with exponent equal to **2**) takes into account the behavior of terms  $s_1$  and  $s_7$  (that can be also equal to zero), while the factor to the right on both sides (that with exponent equal to **4**) takes into account the behavior of the four terms among  $\{s_2, s_3, s_4, s_5, s_6\}$  which are never equal to zero. The one term  $s_k$  which is zero counts as a factor equal to 1 in the product (5).

Hence, the total number of way  $N_1$  in which there could be only a single pair of *consecutive numbers* within the drawn sequence is then given by:

$$N_1 = \frac{1}{84!} \cdot \binom{5}{1} \cdot \frac{d^{84}}{dx^{84}} \left[ \left(\frac{x^{85}-1}{x-1}\right)^2 \left(\frac{x^{85}-1}{x-1}-1\right)^4 \right] \Big|_{x=0} = 164,007,585. \quad (6)$$

Again, the factor  $\frac{1}{84!}$  is the normalization factor needed to simplify the ‘spurious’ numerical coefficient which originates from the exponent of  $x^{84}$  after multiple derivative.

It should be now evident how to calculate the number of possible drawn sequence with *consecutive numbers* originating from two, three, four or all the five numbers  $s_k$  being equal to zero. Let us call  $N_2, N_3, N_4, N_5$  such numbers with respectively two, three, four and all the five numbers  $s_k$  equal to zero. Applying the same reasoning as for eq. (6), one has:

$$N_2 = \frac{1}{84!} \cdot \binom{5}{2} \cdot \frac{d^{84}}{dx^{84}} \left[ \left(\frac{x^{85}-1}{x-1}\right)^2 \left(\frac{x^{85}-1}{x-1}-1\right)^3 \right] \Big|_{x=0} = 20,247,850. \quad (7)$$

$$N_3 = \frac{1}{84!} \cdot \binom{5}{3} \cdot \frac{d^{84}}{dx^{84}} \left[ \left(\frac{x^{85}-1}{x-1}\right)^2 \left(\frac{x^{85}-1}{x-1}-1\right)^2 \right] \Big|_{x=0} = 987,700 \quad (8)$$

$$N_4 = \frac{1}{84!} \cdot \binom{5}{4} \cdot \frac{d^{84}}{dx^{84}} \left[ \left(\frac{x^{85}-1}{x-1}\right)^2 \left(\frac{x^{85}-1}{x-1}-1\right) \right] \Big|_{x=0} = 17,850. \quad (9)$$

$$N_5 = \frac{1}{84!} \cdot \binom{5}{5} \cdot \frac{d^{84}}{dx^{84}} \left[ \left( \frac{x^{85} - 1}{x - 1} \right)^2 \right] \Big|_{x=0} = 85. \quad (10)$$

The numbers  $N_i$  thus count the amount of sequences with  $i$  pairs of consecutive numbers inside (for example, in the sequence  $\{1, 3, 13, 14, 15, 87\}$ , two pairs of consecutive numbers are present,  $\{13, 14\}$  and  $\{14, 15\}$ , and hence this sequence is counted in  $N_2$ ).

Summarizing, the sought number  $N_c$  is thus equal to the sum:

$$N_c = N_1 + N_2 + N_3 + N_4 + N_5 = 185,261,070 \quad (11)$$

and the sought *consecutive numbers* probability is then:

$$\mathcal{C}(6, 90) = \frac{N_c}{N_t} = \frac{185,261,070}{622,614,630} \approx 29.75\%. \quad (12)$$

This outcome says that, on average, nearly one drawing every three results in the *consecutive numbers* phenomenon; it is definitely a quite common result.

A direct comparison between expected and observed *consecutive numbers* frequencies can be easily made taking the data in Table 1 as a statistical sample. The observed frequency amounts to  $\frac{13}{40} \sim 32\%$ , which is slightly higher than that predicted by eq. (12). This is probably due to the smallness of the statistical sample taken into account.

According to the official statistics [4] on all the drawings up to now (spanning from the last days of 1997 to the whole year 2009), the number of *consecutive numbers* occurrences  $N_c$  is equal to 454, while the total number of drawings  $N_t$  amounts to 1507; hence, the observed frequency calculated on the widest possible statistical sample results to be  $\frac{454}{1507} \sim 30.1\%$ . This figure is reasonably close to that predicted by eq. (12).

Another useful result that can be easily derived is the probability  $P(n, M)$  that among a total of  $M$  drawings  $n$  drawings occur with *consecutive numbers*. It is not difficult to realize that  $P(n, M)$  can be expressed as a straightforward binomial distribution with success probability equal to  $\mathcal{C}(6, 90)$ :

$$P(n, M) = \binom{M}{n} \mathcal{C}(6, 90)^n (1 - \mathcal{C}(6, 90))^{M-n}, \quad (13)$$

with mean  $\mu$  and standard deviation  $\sigma$  equal to:

$$\begin{aligned} \mu &= M \cdot \mathcal{C}(6, 90) \\ \sigma &= \sqrt{M \cdot \mathcal{C}(6, 90) \cdot (1 - \mathcal{C}(6, 90))} \end{aligned}$$

Consecutive numbers categories (see text for the meaning of $N_i$ )	Observed occurrences [4]	Expected occurrences (eq. 13) $\mu \pm \sigma$
$N_1$	396	$\sim 397 \pm 17$
$N_2$	53	$\sim 49 \pm 7$
$N_3$	5	$\sim 2 \pm 2$
$N_4$	0	$\sim 0$
$N_5$	0	$\sim 0$
$N_c$	454	$448 \pm 18$

Table 2: Detailed comparison between observed [4] and expected (eq. (13)) occurrences of *consecutive numbers* strings. As explained in Section 2,  $N_i$  stands for the number of sequence with  $i$  pairs of consecutive numbers (e.g. a sequence like  $\{1, 3, 13, 14, 15, 87\}$  has two pairs of consecutive numbers,  $\{13, 14\}$  and  $\{14, 15\}$ ). The observed occurrences are counted among all the 1507 drawings from December 7, 1997, up to the whole year 2009. The expected occurrences are calculated with the mean and the standard deviation of the binomial distribution (eq. (13)), with success probability equal to  $\frac{N_i}{N_t}$ .

In the case of  $M = 1507$  (namely the number of all the *SuperEnalotto* drawings to date) the mean  $\mu$  of the distribution (13) is equal to 448 and the standard deviation  $\sigma$  is nearly equal to 18. Thus, the observed number of drawings with *consecutive numbers* strings, 454, is well within the expected one  $448 \pm 18$ . Table 2 contains also a detailed comparison between observed [4] and expected (eq. (13)) occurrences of *consecutive numbers* strings for each category  $N_1$ ,  $N_2$ ,  $N_3$ ,  $N_4$ ,  $N_5$ .

### 3 Concluding Remarks

In this paper a way has been suggested for calculating the probability of *consecutive numbers* strings within a sequence of  $n$  numbers randomly drawn (without replacement) among the set of the first  $N$  consecutive numbers, with  $N \gg n$ .

An explicit derivation has been carried out for the special case of *SuperEnalotto*, nowadays the most famous lottery in Italy, with  $N = 90$  and  $n = 6$ . It turned out that, on average, one every three drawings presents one or more *consecutive numbers* strings inside: *a posteriori*, this is obviously not surprising; *a priori*, on the other hand, it admittedly appears so, at least to the author.

One reasonably expects that if the ratio  $n/N$  increases, then the *consec-*

*utive numbers* probability will become higher. In the limit, it is not difficult to prove that if  $n \geq \frac{N+2}{2}$ , for  $N$  even, or if  $n \geq \frac{N+3}{2}$ , for  $N$  odd, the probability is equal to 1. This comes from a direct application of the Pigeonhole Principle.

One also may think to use the results obtained in Section 2, along with some statistical goodness-to-fit tests to quantitatively compare observed and expected occurrences, as a quality control on the lottery operators activity. Similar kind of checks are now common against accounting data and tax frauds, e.g. using the first-digits distribution or Benford's law [5].

Artifactual, man-made number sequences usually show a lower occurrence of *consecutive numbers* strings: commonly, people tend to severely underestimate the presence of *consecutive numbers* strings within randomly generated number sequences.

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